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ON AN ANALYTIC APPROACH TO THREE-DIMENSIONAL CONTACT PROBLEMS OF ELASTICITY THEORY*

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A special kernel approximation is applied to the fundamental two-dimensional integral equation of the contact problem about the frictionless indentation of a rigid stamp into an elastic half-space whereupon it is successfully reduced to a form containing just one-dimensional singular Cauchy-type integrals for a broad class of contact domains. The idea of the method is borrowed from the theory of finite span wings. In the case of a rectangular contact domain the equation obtained decomposes into two one-dimensional integro-differential equations. Considered as examples are the cases of a square stamp and a rectangular stamp with the ratio 1/2 between the sides. Numerical results are compared with those from papers in which numerical methods of solving the problem under consideration were used.

1. The contact problem for an elastic half-space under the assumption of no friction in the contact domain will reduce to the solution of a two-dimensional Fredholm integral equation of the first kind /1/:

$$\iint_S \frac{p(u,v)\,du\,dv}{\sqrt{(x-u)^2+(y-v)^2}} = \frac{\pi E}{1-\nu^2} W(x,y), \quad (x,y)\in S \tag{1.1}$$

Here E is the Young's modulus, ν is the Poisson's ratio, $W(x,y)$ is a known function governing the shape of the base surface of the stamp, S is the contact domain, and $p(x,y)$ is the unknown contact pressure.

A fundamental two-dimensional integral equation /2/

$$\frac{\partial}{\partial y}\iint_S \frac{p(u,v)}{y-v}\left[1-\frac{\sqrt{(x-u)^2+(y-v)^2}}{x-u}\right] du\,dv = -F(x,y), \quad (x,y)\in S \tag{1.2}$$

is also known in finite-span wing theory.

To solve it, Laidlaw, and G.V. Sobolev independently, proposed the following approximation of the radical /2/:

$$R=\sqrt{(x-u)^2+(y-v)^2}\approx |x-u|+|y-v| \tag{1.3}$$

which generalizes both the theory of wings of large span ($R\approx|y-v|$) and the theory of small span wings ($R\approx|x-u|$). The approximation (1.3) is evidently most effective for not very convex domains S, for domains not containing points quite remote from the coordinate axes. Since the square root in (1.2) is in the numerator, the approximation (1.3) permits immediate separation of the integral operators acting on the variables x and y. An analogous approximation of the kernel in (1.1) for the contact problem does not permit direct achievement of this goal. We will show that this difficulty can be overcome after a number of special manipulations.

We apply the approximation (1.3) to (1.1), and we then act on it with the operator $\partial^2/\partial x^2-\partial^2/\partial y^2$ (the derivatives are understood in the generalized sense /3–5/). We take into account the relations

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{1}{|x-u|+|y-v|}=$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{2}{\pi}\iint_0^\infty \frac{\cos\xi(x-u)\cos\eta(y-v)}{\xi+\eta}\,d\xi\,d\eta=$$

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$$\frac{2}{\pi}\left(\int_0^\infty \eta\cos\eta(y-v)\,d\eta\int_0^\infty \cos\xi(x-u)\,d\xi-\right.$$

$$\left.\int_0^\infty \xi\cos\xi(x-u)\,d\xi\int_0^\infty \cos\eta(y-v)\,d\eta\right),\qquad \int_0^\infty \cos\xi x\,d\xi=\pi\delta(x).$$

$$\int_0^\infty \xi\cos\xi x\,d\xi=\frac{d}{dx}\int_0^\infty \sin\xi x\,d\xi=\frac{d}{dx}\left(\frac{1}{x}\right)$$

We consequently obtain

$$\frac{\partial}{\partial x}\iint_S \frac{p(u,v)}{x-u}\delta(y-v)\,du\,dv=\frac{\partial}{\partial y}\iint_S \frac{p(u,v)}{y-v}\delta(x-u)\,du\,dv+\frac{\pi E}{2(1-\nu^2)}(W_{yy}-W_{xx}) \tag{1.4}$$

Furthermore, we limit ourselves to an examination of such contact domains for which any segment parallel to one of the coordinate axes, and with its ends within the domain S, belongs entirely to the domain S. For simplicity, we also consider only domains symmetric with respect to the coordinate axes. Let the equation of the domain boundary in the first quadrant be $y=b(x)$ or $x=a(y)$. Then, by using the main property of the delta-function, we finally arrive at the equation

$$\frac{\partial}{\partial x}\int_{-a(y)}^{a(y)}\frac{p(u,y)}{x-u}\,du=\frac{\partial}{\partial y}\int_{-b(x)}^{b(x)}\frac{p(x,v)}{y-v}\,dv+\frac{\pi E}{2(1-\nu^2)}(W_{yy}-W_{xx}) \tag{1.5}$$

Therefore, by using the Laidlaw—Sobolev approximation (1.3), the fundamental integral equation (1.1) is reduced to (1.5) which contains only one-dimensional differentiation operators and one-dimensional singular integral Cauchy-type operators.

2. Let us examine the case of a rectangular stamp with a flat base in more detail. In this case $a(y)\equiv a,\ b(x)\equiv b,\ W(x,y)=W\equiv \text{const}$, and therefore, (1.5) takes on the following form

$$\frac{\partial}{\partial x}\int_{-a}^{a}\frac{p(u,y)}{x-u}\,du=\frac{\partial}{\partial y}\int_{-b}^{b}\frac{p(x,v)}{y-v}\,dv \tag{2.1}$$

We shall seek the solution of (2.1) in the form $p(x,y)=A(x)B(y)$. After the change of variables $y=by',\ x=ax'$, we obtain two one-dimensional equations for the functions $A(x)$ and $B(y)$ from (2.1) (we omit the primes):

$$\frac{d}{dx}\int_{-1}^{1}\frac{A(u)\,du}{x-u}=\mu A(x),\qquad |x|<1 \tag{2.2}$$

$$\frac{d}{dy}\int_{-1}^{1}\frac{B(v)\,dv}{y-v}=\mu\lambda B(y),\qquad |y|<1,\qquad \lambda=\frac{b}{a} \tag{2.3}$$

(μ is a certain constant that should be determined below).

Let us investigate (2.2). The solution of (2.3) is constructed analogously. We note that the function

$$D(x)=\int_0^x A(t)\,dt$$

satisfies the known Prandtl integro-differential equation /6—8/.

The method of trigonometric expansions /6/ applied to (2.2) results in a representation of the function $A(x)$ in a series of orthogonal Chebyshev polynomials of the first kind

$$A(x)=\frac{1}{\sqrt{1-x^2}}\sum_{k=0}^{\infty}a_kT_{2k}(x),\qquad |x|<1 \tag{2.4}$$

after which (2.2) is reduced to an infinite algebraic system in the coefficients a_k /6,8/

$$\frac{\pi^2}{2}a_m-\mu\sum_{k=1}^{\infty}c_{mk}a_k=\mu a_0d_m,\qquad m=1,2,\ldots \tag{2.5}$$

$$c_{mk} = \frac{1}{2k}\left[\frac{1}{4(m+k)^2-1} - \frac{1}{4(m-k)^2-1}\right],$$
$$d_m = -\left[\frac{1}{(2m-1)^2} - \frac{1}{(2m+1)^2}\right]$$

The system (2.5) is constructed such that all the unknown coefficients a_k ($k = 1, 2, \ldots$) are expressed in terms of a_0 after the solution. The rate of decrease in the elements of the matrix $\{c_{mk}\}$ of the system as $m, k \to \infty$, as well as the rate of decrease of the elements of the free terms vector $\{d_m\}$ as $m \to \infty$ are sufficient for quasi-complete regularity of the system for any μ /8/, as well as for the existence and uniqueness of its solution for a value of μ distinct from the eigenvalue of the homogeneous system /9/. We also apply the method of reduction /9/ to solve the system (2.5). If we limit ourselves to just one equation, we obtain

$$a_1 = -a_0 f(\mu), \quad f(\mu) = \frac{8}{9}\mu\left(\frac{\pi^2}{2} - \frac{8}{15}\mu\right)^{-1}$$

and we arrive at an expression for the function $A(x)$ in conformity with (2.4). The solution of (2.3) is found analogously. In sum, the required contact pressure is determined in the form

$$p(x, y) = \frac{a_0 b_0}{\sqrt{(1-x^2)(1-y^2)}}[1 - f(\mu) T_2(x)][1 - f(\lambda\mu) T_2(y)] \tag{2.6}$$

The two constants $a_0 b_0$ and μ are still unknown in this expression. Their appearance is associated with the fact that a second order differentiation operator was applied to the initial equation. And as is known, this expands the class of functions in which the solution of the problem is sought. To extract the unique solution, i.e., to find the quantities $a_0 b_0$ and μ, the representation (2.6) must be substituted into the initial equation (1.1). Multiplying both sides of this equation scalarly by $T_2(x)/\sqrt{1-x^2}$, we arrive at a quadratic equation in μ whose coefficients are one-dimensional improper integrals of Bessel functions and are evaluated by using an electronic computer. Only one of the two values of the parameter μ that are the solution of the quadratic equation, only one will assure the positivity of the contact pressure $p(x, y)$ determined by (2.6) in the whole contact domain. It is the positive and least in absolute value. By knowing the value of μ, we determine the coefficient $a_0 b_0$ after multiplying (1.1) scalarly by $T_0(x)/\sqrt{1-x^2}$.

Specific calculations show that the value of the parameter μ grows as the ratio λ of the sides of the stamp diminishes. Convergence of the series (2.4) is hence degraded, i.e., application of the method of orthogonal polynomials to (2.2) and (2.3) is not effective for a narrow stamp. On the other hand, for sufficiently large values of μ, as follows from (2.6), domains of negative contact pressure, which have no physical meaning, start to appear in the neighborhood of the axes of symmetry. To study the question up to what value is an increase in the parameter μ allowable, the spectral properties of (2.2) must be studied. However, because its equivalent, the Prandtl equation, has been studied so much, it is more convenient to investigate this latter.

After sequential inversion of the Cauchy-type operator and the differentiation operator, the Prandtl equation is converted to an equivalent Fredholm equation of the second kind /10/

$$D(x) = \mu \int_{-1}^{1} K(x, t) D(t)\, dt + \frac{2}{\pi} D(1) \arcsin x, \quad |x| < 1 \tag{2.7}$$

$$K(x, t) = \frac{1}{2\pi^2} \ln \frac{1 - xt + \sqrt{(1-x^2)(1-t^2)}}{1 - xt - \sqrt{(1-x^2)(1-t^2)}}$$

The following /10/ is valid relative to the spectral properties of (2.7): all the characteristic values of the homogeneous equation (2.7) are positive, and form a countable discrete spectrum; the value of the least eigennumber μ_1 obtained from certain integral estimates is 3.637 ± 10^{-3} (it is simple); for $\mu < \mu_1$ the equation (2.7) has a unique continuous solution. It hence follows that the values $\mu < 3.637$ are allowable for the method proposed.

Let us note that the value of μ_1 close to the value $\mu = 3.47$ for which the negative contact pressure domains start to appear when using (2.6).

3. Examples. Square Stamp $(\lambda = 1)$. Here

$$\mu = 0.0923, \; a_0 b_0 = 0.244 \frac{WE}{a(1-\nu^2)}; \quad k = \frac{P(1-\nu^2)}{aWE} = 2.41 \tag{3.1}$$

The two-sided estimates $2.26 < k < 2.81$ /1/ and the more exact $2.26 < k < 2.37$ /11/ were obtained earlier for a dimensionless settling factor in the case of a square stamp. It is seen that the value of k obtained in this paper satisfies the first of these estimates but not the second, however, its relative deviation from the mean value $k = 2.31$ /11/ is 5%. As a comparison, we still present values of k obtained in other papers: $k = 2.300$ (numerical solution /12/), $k = 2.304$*, $k = 2.34$ (in this case the square contact domain was considered as almost circular /13/).

The distribution of dimensionless contact pressure values

$$10^3 \times p(x, y)\, a (1-\nu^2)/WE$$

determined by (2.6) with (3.1) taken into account, is in the lower left side of Table 1. Presented in the upper right side of Table 1 for comparison are analogous values obtained in the paper mentioned in the footnote if we limit ourselves to four coordinate functions of the variational method used there.

Rectangular Stamp with $\lambda = 1/2$ as ratio of the sides. Here

$$\mu = 0.644, \quad a_0 b_0 = 0.172 \frac{WE}{(1-\nu^2)\, b}; \quad k = 1.699 \tag{3.2}$$

For comparison we present values obtained earlier: $k = 1.671$ (see footnote), $k = 1.644$ (numerical solution /14/).

Table 1

y			$x=0$	0.1	0.2	0.3	0.5	0.7	0.8	0.9
0	253		273	274	279	286	315	381	454	624
0.1	254	255		276	280	287	316	383	455	626
0.2	257	259	262		284	292	320	387	460	631
0.3	264	265	269	276		299	328	395	468	641
0.5	289	290	295	302	331		357	425	500	680
0.7	348	349	355	364	398	479		498	579	777
0.8	412	414	420	430	472	567	672		669	889
0.9	564	566	575	589	646	777	920	1260		1170
	$x=0$	0.1	0.2	0.3	0.5	0.7	0.8	0.9		

Table 2

y	$x=0$	0.2	0.6	1.0	1.4	1.8
0	411 404	412 405	422 418	448 450	513 527	773 821
0.1	412 406	413 408	424 420	450 452	515 528	776 822
0.3	426 426	427 428	438 439	465 469	532 543	802 831
0.5	461 475	462 476	473 486	502 512	575 579	867 857
0.7	543 585	545 586	558 593	592 612	678 668	1020 933
0.9	856 980	858 980	879 980	933 984	1070 1020	1610 1300

*Gol'dshtein R.V., Entov V.M., and Zazovskii A.F., Solution of mixed boundary value problems by a direct variational method. Preprint No.78, Inst. Probl. Mekhan., Akad. Nauk SSSR, 1976.

The distribution of dimensionless contact pressure values obtained by means of (2.6) is represented by the odd rows in Table 2; analogous values taken from the paper mentioned in the footnote are given in the even rows.

Let us note that the method developed in this paper and given a foundation by the Laidlaw — Sobolev approximation (1.3) apparently does not permit taking account explicitly of the contact pressure singularity in the neighborhood of the stamp corners.

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